

EXTENSION OF HUYGENS TYPE INEQUALITIES FOR BESSEL AND MODIFIED BESSEL FUNCTIONS

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ABSTRACT. In this paper, new sharpened Huygens type inequalities involving Bessel and modified Bessel functions are established.

keywords: The Bessel functions, The modified Bessel functions, Huygens type inequalities.

1. Introduction

This inequality

$$(1) \quad 2\frac{\sin x}{x} + \frac{\tan x}{x} > 3$$

which holds for all $x \in (0, \pi/2)$ is known in literature as Huygens's inequality [6]. The hyperbolic counterpart of (1) was established in [8] as follows:

$$(2) \quad 2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 3, \quad x > 0.$$

The inequalities (1) and (2) were respectively refined in [6] as

$$(3) \quad 2\frac{\sin x}{x} + \frac{\tan x}{x} > 2\frac{x}{\sin x} + \frac{x}{\tan x} > 3$$

for $0 < x < \frac{\pi}{2}$ and

$$(4) \quad 2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 2\frac{x}{\sinh x} + \frac{x}{\tanh x} > 3, \quad x \neq 0.$$

Recently, in [12], Zhu give some new inequalities of the Huygens type for circular functions, hyperbolic functions, and the reciprocals of circular and hyperbolic functions, as follows:

Theorem A The following inequalities

$$(5) \quad (1-p)\frac{x}{\sin x} + p\frac{x}{\tan x} > 1 > (1-q)\frac{x}{\sin x} + q\frac{x}{\tan x}$$

holds for all $x \in (0, \pi/2)$ if and only if $p \leq 1/3$ and $q \geq 1 - 2/\pi$.

Theorem B The following inequalities

$$(6) \quad (1-p)\frac{\sin x}{x} + p\frac{\tan x}{x} > 1 > (1-q)\frac{\sin x}{x} + q\frac{\tan x}{x}$$

holds for all $x \in (0, \pi/2)$ if and only if $p \geq 1/3$ and $q \leq 0$.

Theorem C The following inequalities

$$(7) \quad (1-p)\frac{\sinh x}{x} + p\frac{\tanh x}{x} > 1 > (1-q)\frac{\sinh x}{x} + q\frac{\tanh x}{x}$$

holds for all $x \in (0, \infty)$ if and only if $p \leq 1/3$ and $q \geq 1$.

Theorem D The following inequalities

$$(8) \quad (1-p)\frac{x}{\sinh x} + p\frac{x}{\tanh x} > 1 > (1-q)\frac{x}{\sinh x} + q\frac{x}{\tanh x}$$

holds for all $x \in (0, \infty)$ if and only if $p \geq 1/3$ and $q \leq 0$.

In this paper, we first give a generalizations of inequalities (5) and (6) to Bessel functions of the first kind and present an conjecture, which may be of interest for further research. Second, we extend and sharpen inequalities (7) and (8) for the modified Bessel functions of the first kind.

2. Lemmas

We begin this section with the following useful lemmas which are needed to completes the proof of the main theorems.

Lemma 1. [7, 2, 9] *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two continuous functions which are differentiable on (a, b) . Further, let $g' \neq 0$ on (a, b) . If $\frac{f'}{g'}$ is increasing (or decreasing) on (a, b) , then the functions $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$ are also increasing (or decreasing) on (a, b) .*

Lemma 2. [10] *Let a_n and b_n ($n = 0, 1, 2, \dots$) be real numbers, and let the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent for $|x| < R$. If $b_n > 0$ for $n = 0, 1, \dots$, and if $\frac{a_n}{b_n}$ is strictly increasing (or decreasing) for $n = 0, 1, 2, \dots$, then the function $\frac{A(x)}{B(x)}$ is strictly increasing (or decreasing) on $(0, R)$.*

3. Extensions of Huygens type inequalities to Bessel functions

In this section, our aim is to extend the inequalities (5) and (6) to Bessel functions of the first kind. For this suppose that $\nu > -1$ and consider the function $\mathcal{J}_\nu : \mathbb{R} \rightarrow (-\infty, 1]$, defined by

$$\mathcal{J}_\nu(x) = 2^\nu \Gamma(\nu + 1) x^{-\nu} J_\nu(x) = \sum_{n \geq 0} \frac{\left(\frac{-1}{4}\right)^n}{(\nu + 1)_n n!} x^{2n},$$

where Γ is the gamma function, $(\nu + 1)_n = \Gamma(\nu + n + 1)/\Gamma(\nu + 1)$ for each $n \geq 0$, is the well-known Pochhammer (or Appell) symbol, and J_ν defined by

$$J_\nu(x) = \sum_{n \geq 0} \frac{(-1)^n (x/2)^{\nu+2n}}{n! \Gamma(\nu + n + 1)},$$

stands for the Bessel function of the first kind of order ν . It is worth mentioning that in particular the function J_ν reduces to some elementary functions, like sine and cosine. More precisely, in particular we have:

$$(9) \quad \mathcal{J}_{-1/2}(x) = \sqrt{\pi/2} x^{1/2} J_{-1/2}(x) = \cos x,$$

$$(10) \quad \mathcal{J}_{1/2}(x) = \sqrt{\pi/2} x^{-1/2} J_{1/2}(x) = \frac{\sin x}{x},$$

respectively, which can verified easily by using the series representation of the function J_ν and of the cosine and sine functions, respectively. Taking into account the relations (9) and (10) as we mentioned above, the inequalities (5) and (6) can be rewritten in terms of $\mathcal{J}_{-1/2}(x)$ and $\mathcal{J}_{1/2}(x)$. For example, using (9) and (10) the inequalities (5) and (6) can be rewritten as

$$(11) \quad (1-p) \frac{1}{\mathcal{J}_{1/2}(x)} + p \frac{\mathcal{J}_{-1/2}(x)}{\mathcal{J}_{1/2}(x)} > 1 > (1-q) \frac{1}{\mathcal{J}_{1/2}(x)} + q \frac{\mathcal{J}_{-1/2}(x)}{\mathcal{J}_{1/2}(x)}$$

and

$$(12) \quad (1-p) \mathcal{J}_{1/2}(x) + p \frac{\mathcal{J}_{1/2}(x)}{\mathcal{J}_{-1/2}(x)} > 1 > (1-q) \mathcal{J}_{1/2}(x) + q \frac{\mathcal{J}_{1/2}(x)}{\mathcal{J}_{-1/2}(x)}$$

and thus it is natural to ask what is the general form of the inequalities (5) and (6) for arbitrary ν .

Our first main result is an extension of inequalities (5) to Bessel functions of the first kind J_ν .

Theorem 1. *Let $\nu > -1$ and let $j_{\nu,1}$ the first positive zero of the Bessel function J_ν of the first kind. Then the Huygens types inequalities*

$$(13) \quad (1-p) \frac{1}{\mathcal{J}_{\nu+1}(x)} + p \frac{\mathcal{J}_\nu(x)}{\mathcal{J}_{\nu+1}(x)} > 1 > (1-q) \frac{1}{\mathcal{J}_{\nu+1}(x)} + q \frac{\mathcal{J}_\nu(x)}{\mathcal{J}_{\nu+1}(x)},$$

holds for all $x \in (0, j_{\nu,1})$ if and only if $p \leq \frac{\nu+1}{\nu+2}$ and $q \geq 1 - \mathcal{J}_\nu(j_{\nu,1})$.

Proof. We define the function $F_\nu(x)$ on $(0, j_{\nu,1})$ by

$$F_\nu(x) = \frac{\frac{1}{\mathcal{J}_{\nu+1}(x)} - 1}{\frac{1}{\mathcal{J}_{\nu+1}(x)} - \frac{\mathcal{J}_\nu(x)}{\mathcal{J}_{\nu+1}(x)}} = \frac{1 - \mathcal{J}_{\nu+1}(x)}{1 - \mathcal{J}_\nu(x)} = \frac{h_{\nu,1}(x)}{h_{\nu,2}(x)},$$

where $f_{\nu,1}(x) = 1 - \mathcal{J}_{\nu+1}(x)$ and $f_{\nu,2}(x) = 1 - \mathcal{J}_\nu(x)$. Now, by again using the differentiation formula

$$(14) \quad \mathcal{J}'_\nu(x) = -\frac{x}{2(\nu+1)} \mathcal{J}_{\nu+1}(x)$$

and the infinite product representation [[11], p. 498]

$$(15) \quad \mathcal{J}_\nu(x) = \prod_{n \geq 1} \left(1 - \frac{x^2}{j_{\nu,n}^2}\right)$$

we obtain that

$$\begin{aligned} \frac{f'_{\nu,1}(x)}{f'_{\nu,2}(x)} &= \frac{(\nu+1)\mathcal{J}_{\nu+2}(x)}{(\nu+2)\mathcal{J}_{\nu+1}(x)} \\ &= 4(\nu+1) \sum_{n \geq 1} \frac{1}{j_{\nu+1,n}^2 - x^2} \end{aligned}$$

So

$$\left(\frac{f'_{\nu,1}(x)}{f'_{\nu,2}(x)} \right)' = 8(\nu+1) \sum_{n \geq 1} \frac{x}{(j_{\nu+1,n}^2 - x^2)^2}.$$

From this, we deduce that the function $\frac{f'_{\nu,1}(x)}{f'_{\nu,2}(x)}$ is increasing on $(0, j_{\nu,1})$. Thus, the function $F_\nu(x)$ is also increasing on $(0, j_{\nu,1})$ by Lemma 2.

In view of $\lim_{x \rightarrow 0^+} F_\nu(x) = \frac{\nu+1}{\nu+2}$ and $\lim_{x \rightarrow j_{\nu,1}} F_\nu(x) = 1 - \mathcal{J}_{\nu+1}(j_{\nu,1})$. With this the proof is complete. \blacksquare

Theorem 2. *Let $-1 < \nu \leq 0$ and let $j_{\nu,1}$ the first positive zero of the Bessel function J_ν of the first kind. Then the Huygens type inequalities*

$$(16) \quad (1-p)\mathcal{J}_{\nu+1}(x) + p \frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} > 1 > (1-q)\mathcal{J}_{\nu+1}(x) + q \frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)}$$

holds for all $x \in (0, j_{\nu,1})$, if and only if, $p \geq \frac{\nu+1}{\nu+2}$ and $q \leq 0$.

Proof. Let $\nu > -1$, consider the function

$$G_\nu(x) = \frac{1 - \mathcal{J}_{\nu+1}(x)}{\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} - \mathcal{J}_{\nu+1}(x)}, \quad 0 < x < j_{\nu,1}.$$

For $0 < x < j_{\nu,1}$, let

$$g_{\nu,1}(x) = 1 - \mathcal{J}_{\nu+1}(x) \text{ and } g_{\nu,2}(x) = \frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_{\nu}(x)} - \mathcal{J}_{\nu+1}(x).$$

From the differentiation formula (14), we get

$$\frac{g'_{\nu,1}(x)}{g'_{\nu,2}(x)} = \frac{1}{1 + \frac{1}{\mathcal{J}_{\nu}(x)} \left(\frac{\nu+2}{\nu+1} \cdot \frac{\mathcal{J}_{\nu+1}^2(x)}{\mathcal{J}_{\nu}(x)\mathcal{J}_{\nu+2}(x)} - 1 \right)} = \frac{1}{1 + \frac{L_{\nu}(x)}{\mathcal{J}_{\nu}(x)}}$$

where

$$L_{\nu}(x) = \frac{\nu+2}{\nu+1} \cdot \frac{\mathcal{J}_{\nu+1}^2(x)}{\mathcal{J}_{\nu}(x)\mathcal{J}_{\nu+2}(x)} - 1.$$

On other hand, using the Turán type inequality [[3], eq. 2.9]

$$(17) \quad \mathcal{J}_{\nu+1}^2(x) - \mathcal{J}_{\nu}(x)\mathcal{J}_{\nu+2}(x) > 0.$$

where $\nu > -1$ and $x \in (-j_{\nu,1}, j_{\nu,1})$, we obtain that the function $L_{\nu}(x)$ is positive on $(0, j_{\nu,1})$.

Elementary calculations reveal that

$$(18) \quad L'_{\nu}(x) = \frac{(\nu+2)x\mathcal{J}_{\nu+1}(x)}{(\nu+1)\mathcal{J}_{\nu}^2(x)\mathcal{J}_{\nu+2}(x)} \left[\frac{\mathcal{J}_{\nu+1}^2(x)}{2(\nu+1)} - \frac{\mathcal{J}_{\nu}(x)\mathcal{J}_{\nu+2}(x)}{\nu+2} \right] + \frac{(\nu+2)x\mathcal{J}_{\nu+1}^2(x)\mathcal{J}_{\nu+2}(x)}{2(\nu+3)(\nu+1)\mathcal{J}_{\nu}(x)\mathcal{J}_{\nu+2}^2(x)}$$

Using the fact that $\mathcal{J}_{\nu+1}(x) \geq \mathcal{J}_{\nu}(x) > 0$ for all $x \in (0, j_{\nu,1})$ and the Turán type inequality (17), we get

$$(19) \quad L'_{\nu}(x) \geq \frac{-\nu x \mathcal{J}_{\nu+1}^3(x)}{2(\nu+1)^2 \mathcal{J}_{\nu}^2(x) \mathcal{J}_{\nu+2}^2(x)} + \frac{(\nu+1)x \mathcal{J}_{\nu+1}^2(x) \mathcal{J}_{\nu+3}(x)}{2(\nu+3)(\nu+1) \mathcal{J}_{\nu}(x) \mathcal{J}_{\nu+2}^2(x)}$$

Thus, we conclude that the function $L_{\nu}(x)$ is increasing on $(0, j_{\nu,1})$ for all $-1 < \nu \leq 0$. Since the function $x \mapsto \mathcal{J}_{\nu}(x)$ is decreasing ([3], Theorem 3) on $(0, j_{\nu,1})$, we gave that the function $\frac{L_{\nu}(x)}{\mathcal{J}_{\nu}(x)}$ is increasing too on $(0, j_{\nu,1})$, as a product of two positives increasing functions. Thus, the function $\frac{g'_{\nu,1}(x)}{g'_{\nu,2}(x)}$ is decreasing on $(0, j_{\nu,1})$. Then, the function

$$G_{\nu}(x) = \frac{g_{\nu,1}(x)}{g_{\nu,2}(x)} = \frac{g_{\nu,1}(x) - g_{\nu,1}(0)}{g_{\nu,2}(x) - g_{\nu,2}(0)}.$$

is decreasing on $(0, j_{\nu,1})$, by Lemma 2,

At the same time, we can write the function $G_{\nu}(x)$ in the following form

$$G_{\nu}(x) = \frac{\mathcal{J}_{\nu}(x)}{\mathcal{J}_{\nu+1}(x)} \cdot F_{\nu}(x).$$

So

$$\lim_{x \rightarrow 0^+} G_{\nu}(x) = F_{\nu}(0) = \frac{\nu+1}{\nu+2} \text{ and } \lim_{x \rightarrow j_{\nu,1}} G_{\nu}(x) = 0,$$

and with this the proof of inequalities (16) is done. ■

Remark 1. Since $j_{-1/2,1} = \frac{\pi}{2}$ we find that the inequalities (13) and (16) is the generalization of inequalities (5) and (6).

Conjecture. The function

$$x \mapsto G_{\nu}(x) = \frac{1 - \mathcal{J}_{\nu+1}(x)}{\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_{\nu}(x)} - \mathcal{J}_{\nu+1}(x)},$$

is decreasing on $(0, j_{\nu,1})$ and $\nu > -1$. If our present conjecture were correct, then this would lead to a extended the inequalities (16).

4. Extensions of the Huygens type inequalities to modified Bessel functions

In this section, we present a generalization of inequalities (7) and (8). For $\nu > -1$ let us consider the function $\mathcal{I}_\nu : \mathbb{R} \rightarrow [1, \infty)$, defined by

$$\mathcal{I}_\nu(x) = 2^\nu \Gamma(\nu + 1) x^{-\nu} I_\nu(x) = \sum_{n \geq 0} \frac{(\frac{1}{4})^n}{(\nu + 1)_n n!} x^{2n},$$

where I_ν is the modified Bessel function of the first kind defined by

$$I_\nu(x) = \sum_{n \geq 0} \frac{(x/2)^{\nu+2n}}{n! \Gamma(\nu + n + 1)}, \text{ for all } x \in \mathbb{R}.$$

It is worth mentioning that in particular we have

$$(20) \quad \mathcal{I}_{-1/2}(x) = \sqrt{\pi/2} x^{1/2} I_{-1/2}(x) = \cosh x,$$

$$(21) \quad \mathcal{J}_{1/2}(x) = \sqrt{\pi/2} x^{1/2} I_{-1/2}(x) = \frac{\sinh x}{x}.$$

Thus, the function I_ν is of special interest in this paper because inequalities (7) and (8) is actually equivalent to

$$(22) \quad (1-p) \mathcal{I}_{1/2}(x) + p \frac{\mathcal{I}_{1/2}(x)}{\mathcal{I}_{-1/2}(x)} > 1 > (1-q) \mathcal{I}_{1/2}(x) + q \frac{\mathcal{I}_{1/2}(x)}{\mathcal{I}_{-1/2}(x)},$$

for all $x \in (0, \infty)$ if and only if $p \leq \frac{-1/2+1}{-1/2+2} = 1/3$ and $q \geq 1$, and

$$(23) \quad (1-p) \frac{1}{\mathcal{I}_{-1/2+1}(x)} + p \frac{\mathcal{I}_{-1/2}(x)}{\mathcal{I}_{-1/2+1}(x)} > 1 > (1-q) \frac{1}{\mathcal{I}_{-1/2+1}(x)} + q \frac{\mathcal{I}_{-1/2}(x)}{\mathcal{I}_{-1/2+1}(x)},$$

for all $x \in (0, \infty)$ if and only if $p \geq \frac{-1/2+1}{-1/2+2} = 1/3$ and $q \leq 0$.

So in view of inequalities (22) and (23) it is natural to ask: what is the analogue of this inequalities for modified Bessel functions of the first kind? In order to answer this question we prove the following results.

Theorem 3. *Let $\nu > -1$, the following inequalities*

$$(24) \quad (1-p) \mathcal{I}_{\nu+1}(x) + p \frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_\nu(x)} > 1 > (1-q) \mathcal{I}_{\nu+1}(x) + q \frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_\nu(x)},$$

holds for all $x \in (0, \infty)$ if and only if $p \leq \frac{\nu+1}{\nu+2}$ and $q \geq 1$.

Proof. Let $\nu > -1$, we define the function H_ν on $(0, \infty)$ by

$$H_\nu(x) = \frac{\mathcal{I}_{\nu+1}(x) - 1}{\mathcal{I}_{\nu+1}(x) - \frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_\nu(x)}} = \frac{\mathcal{I}_{\nu+1}(x)\mathcal{I}_\nu(x) - \mathcal{I}_\nu(x)}{\mathcal{I}_{\nu+1}(x)\mathcal{I}_\nu(x) - \mathcal{I}_{\nu+1}(x)} = \frac{h_{\nu,1}(x)}{h_{\nu,2}(x)},$$

where $h_{\nu,1}(x) = \mathcal{I}_{\nu+1}(x)\mathcal{I}_\nu(x) - \mathcal{I}_\nu(x)$ and $h_{\nu,2}(x) = \mathcal{I}_{\nu+1}(x)\mathcal{I}_\nu(x) - \mathcal{I}_{\nu+1}(x)$. By using the differentiation formula [[11], p. 79]

$$(25) \quad \mathcal{I}'_\nu(x) = \frac{x}{2(\nu+1)} \mathcal{I}_{\nu+1}(x)$$

can easily show that

$$(26) \quad h'_{\nu,1}(x) = \frac{x}{2(\nu+1)} \mathcal{I}_{\nu+1}^2(x) + \frac{x}{2(\nu+2)} \mathcal{I}_{\nu}(x) \mathcal{I}_{\nu+2}(x) - \frac{x}{2(\nu+1)} \mathcal{I}_{\nu+1}(x),$$

and

$$(27) \quad h'_{\nu,2}(x) = \frac{x}{2(\nu+1)} \mathcal{I}_{\nu+1}^2(x) + \frac{x}{2(\nu+2)} \mathcal{I}_{\nu}(x) \mathcal{I}_{\nu+2}(x) - \frac{x}{2(\nu+2)} \mathcal{I}_{\nu+2}(x)$$

Using the Cauchy product

$$(28) \quad I_{\mu}(x) I_{\nu}(x) = \sum_{n \geq 0} \frac{\Gamma(\nu + \mu + 2n + 1) x^{\nu + \mu + 2n}}{2^{\mu + \nu + 2n} \Gamma(n + 1) \Gamma(\nu + \mu + n + 1) \Gamma(\mu + n + 1) \Gamma(\nu + n + 1)}$$

we obtain

$$(29) \quad h'_{\nu,1}(x) = \sum_{n \geq 0} A_n(\nu) x^{2n}$$

and

$$(30) \quad h'_{\nu,2}(x) = \sum_{n \geq 0} B_n(\nu) x^{2n}$$

where

$$(31) \quad A_n(\nu) = \frac{\Gamma(\nu + 1) \left(\Gamma(\nu + 2) \Gamma(2\nu + 2n + 4) - \Gamma(2\nu + n + 3) \Gamma(\nu + n + 3) \right)}{2^{2n+1} \Gamma(n + 1) \Gamma(\nu + n + 2) \Gamma(\nu + n + 3) \Gamma(2\nu + n + 3)}$$

and

$$(32) \quad B_n(\nu) = \frac{\Gamma(\nu + 2) \left(\Gamma(\nu + 1) \Gamma(2\nu + 2n + 4) - \Gamma(2\nu + n + 3) \Gamma(\nu + n + 2) \right)}{2^{2n+1} \Gamma(n + 1) \Gamma(\nu + n + 2) \Gamma(\nu + n + 3) \Gamma(2\nu + n + 3)}$$

Now, we define the sequence $C_n = \frac{A_n}{B_n}$ for $n = 0, 1, \dots$, thus

$$C_n(\nu) = \frac{\Gamma(\nu + 1) \left(\Gamma(\nu + 2) \Gamma(2\nu + 2n + 4) - \Gamma(2\nu + n + 3) \Gamma(\nu + n + 3) \right)}{\Gamma(\nu + 2) \left(\Gamma(\nu + 1) \Gamma(2\nu + 2n + 4) - \Gamma(2\nu + n + 3) \Gamma(\nu + n + 2) \right)}$$

So, for $\nu > -1$ and $n = 0, 1, \dots$, we get

$$(33) \quad \begin{aligned} \frac{C_{n+1}(\nu)}{C_n(\nu)} &= \frac{[\Gamma(\nu + 2) \Gamma(2\nu + 2n + 6) - \Gamma(2\nu + n + 4) \Gamma(\nu + n + 4)]}{[\Gamma(\nu + 1) \Gamma(2\nu + 2n + 6) - \Gamma(2\nu + n + 4) \Gamma(\nu + n + 3)]} \\ &\times \frac{[\Gamma(\nu + 1) \Gamma(2\nu + 2n + 4) - \Gamma(2\nu + n + 3) \Gamma(\nu + n + 2)]}{[\Gamma(\nu + 2) \Gamma(2\nu + 2n + 4) - \Gamma(2\nu + n + 3) \Gamma(\nu + n + 3)]} \\ &= \frac{K_n^1(\nu)}{K_n^2(\nu)} \end{aligned}$$

where

$$\begin{aligned} K_n^1(\nu) &= [\Gamma(\nu + 2) \Gamma(2\nu + 2n + 6) - \Gamma(2\nu + n + 4) \Gamma(\nu + n + 4)] [\Gamma(\nu + 1) \Gamma(2\nu + 2n + 4) - \Gamma(2\nu + n + 3) \Gamma(\nu + n + 2)] \\ &= \underbrace{\Gamma(\nu + 1) \Gamma(\nu + 2) \Gamma(2\nu + 2n + 6) \Gamma(2\nu + 2n + 4)}_{A_1} - \underbrace{\Gamma(\nu + 2) \Gamma(2\nu + 2n + 6) \Gamma(2\nu + n + 3) \Gamma(\nu + n + 2)}_{B_1} \\ &\quad - \underbrace{\Gamma(\nu + 1) \Gamma(2\nu + n + 4) \Gamma(\nu + n + 4) \Gamma(2\nu + 2n + 4)}_{C_1} + \underbrace{\Gamma(2\nu + n + 4) \Gamma(\nu + n + 4) \Gamma(2\nu + n + 3) \Gamma(\nu + n + 2)}_{D_1} \end{aligned}$$

and

$$\begin{aligned}
 K_n^2(\nu) &= [\Gamma(\nu+1)\Gamma(2\nu+2n+6) - \Gamma(2\nu+n+4)\Gamma(\nu+n+3)] [\Gamma(\nu+2)\Gamma(2\nu+2n+4) - \Gamma(2\nu+n+3)\Gamma(\nu+n+3)] \\
 &= \underbrace{\Gamma(\nu+1)\Gamma(\nu+2)\Gamma(2\nu+2n+6)\Gamma(2\nu+2n+4)}_{A_1} - \underbrace{\Gamma(\nu+1)\Gamma(2\nu+2n+6)\Gamma(2\nu+n+3)\Gamma(\nu+n+3)}_{B_2} \\
 &\quad - \underbrace{\Gamma(\nu+2)\Gamma(2\nu+n+4)\Gamma(\nu+n+3)\Gamma(2\nu+2n+4)}_{C_2} + \underbrace{\Gamma(2\nu+n+4)\Gamma^2(\nu+n+3)\Gamma(2\nu+n+3)}_{D_2}
 \end{aligned}$$

Thus

$$K_n^1(\nu) - K_n^2(\nu) = (B_2 - B_1) + (C_2 - C_1) + (D_1 - D_2)$$

A simple calculation we obtain

$$B_2 - B_1 = (n+1)\Gamma(\nu+1)\Gamma(\nu+n+2)\Gamma(2\nu+n+3)\Gamma(2\nu+2n+6)$$

and

$$C_2 - C_1 = -(n+2)\Gamma(\nu+1)\Gamma(2\nu+2n+4)\Gamma(2\nu+n+4)\Gamma(\nu+n+3)$$

and

$$D_1 - D_2 = \Gamma(2\nu+n+4)\Gamma(2\nu+n+3)\Gamma(\nu+n+3)\Gamma(\nu+n+2) \geq 0.$$

Then simple computations lead to

$$\begin{aligned}
 B_2 - B_1 + (C_2 - C_1) &= \Gamma(\nu+1)\Gamma(2\nu+2n+4)\Gamma(2\nu+n+3)\Gamma(\nu+n+2) \left((n+1)(2\nu+2n+5)(2\nu+2n+4) \right. \\
 &\quad \left. - (n+2)(2\nu+2n+3)(\nu+n+2) \right) \\
 &\geq P_n(\nu)(n+1)\Gamma(\nu+1)\Gamma(2\nu+2n+4)\Gamma(2\nu+n+3)\Gamma(\nu+n+2)
 \end{aligned}$$

where

$$\begin{aligned}
 P_n(\nu) &= 2\nu^2 + (4n+11)\nu + 2n^2 + 11n + 14 \\
 &= (\nu+n+2)(2\nu+2n+7) > 0,
 \end{aligned}$$

for all $\nu > -1$ and $n \in \mathbb{N}$. Therefore, the sequence $(C_n)_n$ is increasing, we obtain that the function $\frac{h'_{\nu,1}(x)}{h'_{\nu,2}(x)}$ is increasing on $(0, \infty)$ too (by Lemma 1). Thus $H_\nu(x) = \frac{h_{\nu,1}(x) - h_{\nu,1}(0^+)}{h_{\nu,2}(x) - h_{\nu,2}(0^+)}$ is also increasing on $(0, \infty)$ by Lemma 2. So,

$$\lim_{x \rightarrow 0^+} H_\nu(x) = C_0(\nu) = \frac{\nu+1}{\nu+2},$$

and using the asymptotic formula [[1], p. 377]

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}} \left[1 - \frac{4\nu^2 - 1}{1!(8x)} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2!(8x)^2} - \dots \right]$$

which holds for large values of x and for fixed $\nu > -1$, we obtain

$$\lim_{x \rightarrow \infty} H_\nu(x) = 1.$$

So the proof of Theorem 3 is complete. ■

Theorem 4. Let $\nu > -1$, the following inequalities

$$(34) \quad (1-p) \frac{1}{\mathcal{I}_{\nu+1}(x)} + p \frac{\mathcal{I}_\nu(x)}{\mathcal{I}_{\nu+1}(x)} > 1 > (1-q) \frac{1}{\mathcal{I}_{\nu+1}(x)} + q \frac{\mathcal{I}_\nu(x)}{\mathcal{I}_{\nu+1}(x)},$$

holds for all $x \in (0, \infty)$ if and only if $p \geq \frac{\nu+1}{\nu+2}$ and $q \leq 0$.

Proof. Let $\nu > -1$ and $x \in (0, \infty)$, we define the function $\Phi_\nu(x)$ by

$$(35) \quad \Phi_\nu(x) = \frac{\frac{1}{\mathcal{I}_{\nu+1}(x)} - 1}{\frac{1}{\mathcal{I}_{\nu+1}(x)} - \frac{\mathcal{I}_\nu(x)}{\mathcal{I}_{\nu+1}(x)}} = \frac{1 - \mathcal{I}_{\nu+1}(x)}{1 - \mathcal{I}_\nu(x)} = \frac{\varphi_{\nu,1}(x)}{\varphi_{\nu,2}(x)},$$

where $\varphi_{\nu,1}(x) = 1 - \mathcal{I}_{\nu+1}(x)$ and $\varphi_{\nu,2}(x) = 1 - \mathcal{I}_\nu(x)$. By again using the differentiation formula (25) we get

$$(36) \quad \frac{\varphi'_{\nu,1}(x)}{\varphi'_{\nu,2}(x)} = \frac{\nu+1}{\nu+2} \cdot \frac{\mathcal{I}_{\nu+2}(x)}{\mathcal{I}_{\nu+1}(x)} = \frac{\sum_{n=0}^{\infty} a_n(\nu) x^{2n}}{\sum_{n=0}^{\infty} b_n(\nu) x^{2n}},$$

where $a_n(\nu) = \frac{\Gamma(\nu+2)}{2^{2n}\Gamma(n+1)\Gamma(\nu+n+3)}$ and $b_n(\nu) = \frac{\Gamma(\nu+1)}{2^{2n}\Gamma(n+1)\Gamma(\nu+n+2)}$. Let

$$c_n(\nu) = \frac{a_n(\nu)}{b_n(\nu)} = \frac{\nu+1}{\nu+n+2}, \text{ for } n = 0, 1, \dots$$

We conclude that $c_n(\nu)$ is decreasing for $n = 0, 1, \dots$ and $\frac{g'_1(x)}{g'_2(x)}$ is decreasing on $(0, \infty)$ by Lemma 2. Thus

$$\Phi_\nu(x) = \frac{\varphi_{\nu,1}(x)}{\varphi_{\nu,2}(x)} = \frac{\varphi_{\nu,1}(x) - \varphi_{\nu,1}(0)}{\varphi_{\nu,2}(x) - \varphi_{\nu,2}(0)},$$

is decreasing on $(0, \infty)$ by Lemma 1. Furthermore,

$$\lim_{x \rightarrow 0^+} \Phi_\nu(x) = c_0(\nu) = \frac{\nu+1}{\nu+2},$$

and

$$\lim_{x \rightarrow \infty} \Phi_\nu(x) = 0.$$

Alternatively, inequality (34) can be proved by using the Mittag-Leffler expansion for the modified Bessel functions of first kind, which becomes [[4], Eq. 7.9.3]

$$(37) \quad \frac{I_{\nu+1}(x)}{I_\nu(x)} = \sum_{n=1}^{\infty} \frac{2x}{j_{\nu,n}^2 + x^2},$$

where $0 < j_{\nu,1} < j_{\nu,2} < \dots < j_{\nu,n} < \dots$, are the positive zeros of the Bessel function J_ν , we obtain that

$$\frac{g'_1(x)}{g'_2(x)} = 2(\nu+1) \frac{I_{\nu+2}(x)}{x I_{\nu+1}(x)} = 4(\nu+1) \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2 + x^2}.$$

■

Clearly,

$$\left(\frac{g'_1(x)}{g'_2(x)} \right)' = -8(\nu+1) \sum_{n=1}^{\infty} \frac{x}{(x^2 + j_{\nu,n}^2)^2},$$

for all $x > 0$ and $\nu > -1$, which implies that $G_\nu(x)$ is decreasing for all $\nu > -1$. On the other hand, using the Rayleigh formula [[11], p. 502]

$$(38) \quad \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2} = \frac{1}{4(\nu+1)}.$$

we get

$$\lim_{x \rightarrow 0^+} G_\nu(x) = \frac{\nu+1}{\nu+2}.$$

So, the proof of Theorem 4 is complete.

Remark 2. Since $\mathcal{I}_{-\frac{1}{2}}(x) = \cosh x$ and $\mathcal{I}_{\frac{1}{2}}(x) = \frac{\sinh x}{x}$, we find that the inequalities (24) and (34) is the generalization of inequalities (7) and (8).

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